Characterizing Relative Frame Definability in Team Semantics via the Universal Modality*

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Abstract. Let $\mathcal{ML}(\boxed{\mathsf{u}}^+)$ denote the fragment of modal logic extended with the universal modality in which the universal modality occurs only positively. We characterise the relative definability of $\mathcal{ML}(\boxed{\mathsf{u}}^+)$ relative to finite transitive frames in the spirit of the well-known Goldblatt–Thomason theorem. We show that a class $\mathbb{F}$ of finite transitive frames is definable in $\mathcal{ML}(\boxed{\mathsf{u}}^+)$ relative to finite transitive frames if and only if $\mathbb{F}$ is closed under taking generated subframes and bounded morphic images. In addition, we study modal definability in team-based logics. We study (extended) modal dependence logic, (extended) modal inclusion logic, and modal team logic. With respect to global model definability we obtain a trichotomy and with respect to frame definability a dichotomy. As a corollary we obtain relative Goldblatt–Thomason -style theorems for each of the logics listed above.

1 Introduction

Team semantics was introduced by Hodges [15] in the context of the so-called *independence-friendly logic* of Hintikka and Sandu [14]. The fundamental idea behind team semantics is crisp. The idea is to shift from single assignments to sets of assignments as the satisfying elements of formulas. Väänänen [19] adopted team semantics as the core notion for his *dependence logic*. The syntax of first-order dependence logic extends the syntax of first-order logic by novel atomic formulas called dependence atoms. The intuitive meaning of the dependence atom $=(x_1, \ldots, x_n, y)$ is that inside a team the value of y is functionally determined by the values of $x_1, \ldots, x_n$. After the introduction of dependence logic in 2007 the study of related logics with team semantics has boomed. One of the most important developments in the area of team semantics was the introduction of *independence logic* [10] in which dependence atoms of dependence logic

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are replaced by *independence atoms*. Soon after, Galliani [5] showed that independence atoms can be further analysed, and alternatively expressed, in terms of inclusion and exclusion atoms.

Concurrently a vibrant research on modal and propositional logics with team semantics has emerged. In the context of modal logic, any subset of the domain of a Kripke model is called a team. In modal team semantics, formulas are evaluated with respect to team-pointed Kripke models. The study of *modal dependence logic* was initiated by Väänänen [20] in 2008. Shortly after, *extended modal dependence logic* ($\mathcal{EMDL}$) was introduced by Ebbing et al. [4] and *modal independence logic* by Kontinen et al. [16]. The focus of the research has been in the computational complexity and expressive power. Hella et al. [11] established that exactly the properties of teams that have the so-called *empty team property*, are *downward closed* and closed under the so-called team *k-bisimulation*, for some finite k, are definable in $\mathcal{EMDL}$. Kontinen et al. [17] have shown that exactly the properties of teams that are closed under the team k-bisimulation are definable in the so-called *modal team logic*, whereas Hella and Stumf established [12] that the so-called *extended modal inclusion logic* is characterised by the empty team property, union closure, and closure under team k-bisimulation. See the survey [3] for a detailed exposition on the expressive power and computational complexity of related logics.

The study of frame definability in the team semantics context was initiated by Sano and Virtema [18]. Let $\mathcal{ML}(\boxed{\sqcup}^+)$ denote the syntactic fragment of modal logic with universal modality in which the universal modality occurs only positively. Sano and Virtema established a surprising connection between $\mathcal{ML}(\boxed{\sqcup}^+)$ and particular team-based modal logics and gave a Goldblatt–Thomason -style theorem for the logics in question. They showed that with respect to frame definability $\mathcal{ML}(\boxed{\sqcup}^+)$, $\mathcal{MDL}$ and $\mathcal{EMDL}$ coincide. Moreover, they established that an elementary class of Kripke frames is definable in $\mathcal{ML}(\boxed{\sqcup}^+)$ (and thus in $\mathcal{MDL}$ and $\mathcal{EMDL}$) if and only if it is closed under taking generated subframes and bounded morphic images, and reflects ultrafilter extensions and finitely generated subframes.

Since most familiar modal logics enjoy the finite model property, one may wonder if we can restrict our attention to classes of finite frames for characterizing modal definability. For basic modal logic this was done in [1]. It is immediate to see that the reflection of ultrafilter extensions should be redundant under such restriction because ultrafilter extensions of finite frames are just those frames themselves. Interestingly, a modally undefinable property sometimes becomes definable within a suitable class of finite frames. A first-order condition of irreflexivity (for any w, wRw fails) of the accessibility relation is known to be undefinable by a set of modal formulas, since the condition violates the closure of a modally definable class under surjective bounded morphisms (consider a bounded morphism sending a frame of two symmetric points to a frame of a single reflexive point). It is, however, also known that irreflexivity becomes definable within the class of finite transitive frames by the Loeb axiom $\Box(\Box p \to p) \to \Box p$.

Such phenomena motivate us to study relative definability also in the context of team-based modal logics.

In this paper, we provide Goldblatt–Thomason -style theorem for the relative definability of $\mathcal{ML}(\boxed{\mathrm{u}}^+)$ relative to finite transitive frames in the spirit of [1] with the help of Jankov-Fine formulas (cf. [2, Theorem 3.21]). We show that a class $\mathbb{F}$ of finite transitive frames is definable in $\mathcal{ML}(\boxed{\mathrm{u}}^+)$ relative to finite transitive frames if and only if $\mathbb{F}$ is closed under taking generated subframes and bounded morphic images. In addition, we study modal definability in team-based logics. We study (extended) modal dependence logic, (extended) modal inclusion logic, and modal team logic. We obtain strict hierarchies with respect to both global model definability and frame definability.

2 Modal Logic with Universal Modality

In this section, we introduce modal logic with universal modality and give some basic definitions and results concerning frame definability. In team-based logics it is customary to define the syntax in negation normal form, that is to assume that negations occur only in front of proposition symbols. This is due to the fact that the team semantics negation, that corresponds to the negation used in Kripke semantics, is not the contradictory negation of team semantics. Since in this article we consider extensions of modal logic in the framework of team semantics, we define the syntax of modal logic also in negation normal form.

Let Φ be a set of atomic propositions. The set of formulas for *modal logic* $\mathcal{ML}(\Phi)$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi, \quad \text{where } p \in \Phi.$$

The syntax of *modal logic with universal modality* $\mathcal{ML}(\boxed{\mathrm{u}})(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the grammar rules

$$\varphi ::= \boxed{\mathrm{u}}\,\varphi \mid \Diamond_{\mathrm{u}}\,\varphi.$$

The syntax of *modal logic with positive universal modality* $\mathcal{ML}(\boxed{\mathrm{u}}^+)(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the grammar rule $\varphi ::= \boxed{\mathrm{u}}\,\varphi$. As usual, if the underlying set Φ of atomic propositions is clear from the context, we drop "(Φ)" and just write $\mathcal{ML}$, $\mathcal{ML}(\boxed{\mathrm{u}})$, etc. We also use the shorthands $\neg\varphi$, $\varphi \rightarrow \psi$, and $\varphi \leftrightarrow \psi$. By $\neg\varphi$ we denote the formula that can be obtained from $\neg\varphi$ by pushing all negations to the atomic level, and by $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$, we denote $\neg\varphi \vee \psi$ and $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, respectively.

A (Kripke) *frame* is a pair $\mathfrak{F} = (W, R)$ where W, called the *domain* of $\mathfrak{F}$, is a non-empty set and $R \subseteq W \times W$ is a binary relation on W. By $\mathbb{F}_{\mathrm{all}}$, we denote the class of all frames. We use $|\mathfrak{F}|$ to denote the domain of the frame $\mathfrak{F}$. A (Kripke) *Φ-model* is a tuple $\mathfrak{M} = (W, R, V)$, where (W, R) is a frame and $V : \Phi \rightarrow \mathcal{P}(W)$ is a valuation of the proposition symbols. By $\mathbb{M}_{\mathrm{all}}(\Phi)$, we denote the class of all Φ-models. The semantics of modal logic, i.e., the *satisfaction relation* $\mathfrak{M}, w \Vdash \varphi$,

is defined via pointed Φ-*models* as usual. For the universal modality $\boxed{\mathrm{u}}$ and its dual $\Diamond\!\!\!\!\!\!\mathrm{u}$, we define

$$\begin{aligned}
\mathfrak{M}, w \Vdash \boxed{\mathrm{u}}\,\varphi \quad &\Leftrightarrow \quad \mathfrak{M}, v \Vdash \varphi, \text{ for every } v \in W,\\
\mathfrak{M}, w \Vdash \langle\mathrm{u}\rangle\,\varphi \quad &\Leftrightarrow \quad \mathfrak{M}, v \Vdash \varphi, \text{ for some } v \in W.
\end{aligned}$$

A formula set Γ is *valid in a model* $\mathfrak{M} = (W, R, V)$ (notation: $\mathfrak{M} \Vdash \Gamma$), if $\mathfrak{M}, w \Vdash \varphi$ holds for every $w \in W$ and every $\varphi \in \Gamma$. When Γ is a singleton $\{\,\varphi\,\}$, we simply write $\mathfrak{M} \Vdash \varphi$.

Below we assume only that the logics $\mathcal{L}(\Phi)$ and $\mathcal{L}'(\Phi)$ are such that the global satisfaction relation for Kripke models (i.e., $\mathfrak{M} \Vdash \varphi$) is defined. A set Γ of $\mathcal{L}(\Phi)$-formulas is *valid in a frame* $\mathfrak{F}$ (written: $\mathfrak{F} \Vdash \Gamma$) if $(\mathfrak{F}, V) \Vdash \varphi$ for every valuation $V : \Phi \to \mathcal{P}(W)$ and every $\varphi \in \Gamma$. A set Γ of $\mathcal{L}(\Phi)$-formulas is *valid in a class* $\mathbb{F}$ *of frames* (written: $\mathbb{F} \Vdash \Gamma$) if $\mathfrak{F} \Vdash \Gamma$ for every $\mathfrak{F} \in \mathbb{F}$. Given a set Γ of $\mathcal{L}(\Phi)$-formulas, $\mathbb{FR}(\Gamma) := \{\, \mathfrak{F} \in \mathbb{F}_{\mathrm{all}} \mid \mathfrak{F} \Vdash \Gamma \,\}$ and $\mathrm{Mod}(\Gamma) := \{\, \mathfrak{M} \in \mathbb{M}_{\mathrm{all}}(\Phi) \mid \mathfrak{M} \Vdash \Gamma \,\}$. We say that Γ *defines* the class $\mathbb{F}$ of frames and the class $\mathbb{C}$ of models, if $\mathbb{F} = \mathbb{FR}(\Gamma)$ and $\mathbb{C} = \mathrm{Mod}(\Gamma)$, respectively. When Γ is a singleton $\{\,\varphi\,\}$, we simply say that φ defines the class $\mathbb{F}$ (or $\mathbb{C}$). A class $\mathbb{F}$ of frames (models) is $\mathcal{L}(\Phi)$-*definable* if there exists a set Γ of $\mathcal{L}(\Phi)$-formulas such that $\mathbb{FR}(\Gamma) = \mathbb{F}$ ($\mathrm{Mod}(\Gamma) = \mathbb{F}$).

It was shown in [18] that with respect to frame definability, we have that $\mathcal{ML} < \mathcal{ML}(\boxed{\mathrm{u}}^+) < \mathcal{ML}(\boxed{\mathrm{u}})$. Moreover the frame definability of each of the mentioned logics have been characterised with respect to first-order definable frame classes. For the characterisations the notions of *disjoint unions*, *generated subframes*, *bounded morphisms*, and *ultrafilter extensions* are required. Definitions for these constructions can be found, e.g., in [2], and in Appendix B.

The following results were proved for $\mathcal{ML}$ by Goldblatt and Thomason [7], for $\mathcal{ML}(\boxed{\mathrm{u}}^+)$ by Sano and Virtema [18], and for $\mathcal{ML}(\boxed{\mathrm{u}})$ by Goranko and Passy [9]. A frame class $\mathbb{F}$ *reflects* finitely generated subframes whenever it is the case for all frames $\mathfrak{F}$ that, if every finitely generated subframe of $\mathfrak{F}$ is in $\mathbb{F}$, then $\mathfrak{F} \in \mathbb{F}$.

Theorem 1 (Goldblatt–Thomason theorems for $\mathcal{ML}$, $\mathcal{ML}(\boxed{\mathrm{u}}^+)$ & $\mathcal{ML}(\boxed{\mathrm{u}})$).
(i) An elementary frame class is $\mathcal{ML}$-definable if and only if it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.
(ii) An elementary frame class is $\mathcal{ML}(\boxed{\mathrm{u}}^+)$-definable if and only if it is closed under taking generated subframes and bounded morphic images, and reflects ultrafilter extensions and finitely generated subframes.
(iii) An elementary frame class is $\mathcal{ML}(\boxed{\mathrm{u}})$-definable if and only if it is closed under taking bounded morphic images and reflects ultrafilter extensions.

3 Finite Goldblatt-Thomason-style Theorem for Relative Modal definability with Positive Universal Modality

Given a class $\mathbb{G}$ of frames, we say that a set of formulas *defines* a class $\mathbb{F}$ of frames within $\mathbb{G}$ if, for all frames $\mathfrak{F} \in \mathbb{G}$, the equivalence: $\mathfrak{F} \Vdash \varphi \Leftrightarrow \mathfrak{F} \in \mathbb{F}$ holds. A frame $\mathbb{F} = (W, R)$ is called *finite* whenever W is a finite set and *transitive*

whenever R is a transitive relation. In what follows, let $\mathbb{F}_{\text{fintra}}$ be the class of all finite transitive frames and $\mathbb{F}_{\text{fin}}$ the class of all finite frames.

With the help of frame constructions such as bounded morphic images, disjoint unions, generated subframes, we first review the existing characterisations of relative $\mathcal{ML}$- and $\mathcal{ML}(\boxed{u})$-definability within the class of finite transitive frames. We then give a novel characterisation of relative $\mathcal{ML}(\boxed{u}^+)$-definability again within the class of finite transitive frames.

Theorem 2 (Finite Goldblatt–Thomason Theorems for $\mathcal{ML}$ [1] & $\mathcal{ML}(\boxed{u})$ [6]).

1. *A class of finite transitive frames is $\mathcal{ML}$-definable within the class $\mathbb{F}_{\text{fintra}}$ of all finite transitive frames if and only if it is closed under taking bounded morphic images, generated subframes, and disjoint unions.*
2. *A class of finite frames is $\mathcal{ML}(\boxed{u})$-definable within the class $\mathbb{F}_{\text{fin}}$ of all finite frames if and only if it is closed under taking bounded morphic images.*

In order to show the corresponding characterisation of relative definability in $\mathcal{ML}(\boxed{u}^+)$, a variant of the Jankov-Fine formula is defined.

Definition 1. *Let $\mathfrak{F} = (W, R)$ be a finite transitive frame. Put $W := \{ w_0, \dots, w_n \}$. Associate a new proposition variable p_{w_i} with each w_i and define $\Box^+\varphi := \Box\varphi \wedge \varphi$. The Jankov-Fine formula $\varphi_{\mathfrak{F},w_i}$ at w_i is defined as the conjunction of all the following formulas:*

1. p_{w_i}
2. $\Box(p_{w_0} \vee \cdots \vee p_{w_n})$.
3. $\bigwedge \{ \Box^+(p_{w_i} \to \neg p_{w_j}) \mid w_i \neq w_j \}$.
4. $\bigwedge \{ \Box^+(p_{w_i} \to \Diamond p_{w_j}) \mid (w_i, w_j) \in R \}$.
5. $\bigwedge \{ \Box^+(p_{w_i} \to \neg\Diamond p_{w_j}) \mid (w_i, w_j) \notin R \}$.

The Jankov-Fine formula $\varphi_{\mathfrak{F}}$ is defined as $\bigvee_{w \in W} \boxed{u}\, \neg\varphi_{\mathfrak{F},w}$.

We note that the Jankov-Fine formula $\varphi_{\mathfrak{F},w_i}$ at w_i is an $\mathcal{ML}$-formula and thus the Jankov-Fine formula $\varphi_{\mathfrak{F}}$ is an $\mathcal{ML}(\boxed{u}^+)$-formula.

Lemma 1 (For a proof, see Appendix A). *Let $\mathfrak{F} = (W, R)$ be a finite transitive frame. For any transitive frame $\mathfrak{G}$, the following are equivalent:*

(i) *the Jankov-Fine formula $\varphi_{\mathfrak{F}}$ is not valid in $\mathfrak{G}$,*
(ii) *there is a finite set $Y \subseteq |\mathfrak{G}|$ such that $\mathfrak{F}$ is a bounded morphic image of $\mathfrak{G}_Y$, where $\mathfrak{G}_Y$ is the subframe of $\mathfrak{G}$ generated by Y.*

Theorem 3. *For every class $\mathbb{F}$ of finite transitive frames, the following are equivalent:*

(i) $\mathbb{F}$ *is $\mathcal{ML}(\boxed{u}^+)$-definable within $\mathbb{F}_{\text{fintra}}$.*
(ii) $\mathbb{F}$ *is closed under taking generated subframes and bounded morphic images.*

Proof. The direction from (i) to (ii) is easy to establish, so we focus on the converse direction. Assume (ii). Define $\mathrm{Log}(\mathbb{F}) = \{\, \varphi \in \mathcal{ML}(\boxplus^{+}) \mid \mathbb{F} \Vdash \varphi \,\}$. We show that $\mathrm{Log}(\mathbb{F})$ defines $\mathbb{F}$ within $\mathbb{F}_{\mathrm{fintra}}$. Fix any finite and transitive frame $\mathfrak{F} \in \mathbb{F}_{\mathrm{fintra}}$. In what follows, we show the following equivalence:

$$\mathfrak{F} \in \mathbb{F} \iff \mathfrak{F} \Vdash \mathrm{Log}(\mathbb{F}).$$

The left-to-right direction is immediate, so we concentrate on the converse direction. Assume $\mathfrak{F} \Vdash \mathrm{Log}(\mathbb{F})$. Since $\mathfrak{F}$ is finite and transitive, let us take the Jankov-Fine formula $\varphi_{\mathfrak{F}}$. Since $\varphi_{\mathfrak{F}}$ is not valid in $\mathfrak{F}$, $\varphi_{\mathfrak{F}} \notin \mathrm{Log}(\mathbb{F})$. Thus there is a *transitive* frame $\mathfrak{G} \in \mathbb{F}$ (recall that $\mathbb{F}$ is a class of transitive frames) such that $\varphi_{\mathfrak{F}}$ is not valid in $\mathfrak{G}$. By Lemma 1, there is a finite set $Y \subseteq |\mathfrak{G}|$ such that $\mathfrak{F}$ is a bounded morphic image of $\mathfrak{G}_Y$. Since $\mathfrak{G} \in \mathbb{F}$, $\mathfrak{G}_Y \in \mathbb{F}$ by $\mathbb{F}$'s closure under generated subframes. It follows from $\mathbb{F}$'s closure under bounded morphic images that $\mathfrak{F} \in \mathbb{F}$, as desired. $\square$

4 Modal Logics with Team Semantics

In this section we define the team-based modal logics that are relevant for this paper. We survey basic properties and known result concerning expressive power.

4.1 Basic notions of team semantics

A subset T of the domain of a Kripke model $\mathfrak{M}$ is called *a team of* $\mathfrak{M}$. Before we define the so-called *team semantics* for $\mathcal{ML}$, let us first introduce some notation that makes defining the semantics simpler.

Definition 2. *Let $\mathfrak{M} = (W, R, V)$ be a model and T and S teams of $\mathfrak{M}$. Define*

$$R[T] := \{w \in W \mid \exists v \in T(vRw)\} \text{ and } R^{-1}[T] := \{w \in W \mid \exists v \in T(wRv)\}.$$

For teams T and S of $\mathfrak{M}$, we write $T[R]S$ if $S \subseteq R[T]$ and $T \subseteq R^{-1}[S]$.

Thus, $T[R]S$ holds if and only if for every $w \in T$ there exists some $v \in S$ such that wRv, and for every $v \in S$ there exists some $w \in T$ such that wRv. The team semantics for $\mathcal{ML}$ is defined as follows. We use the symbol "$\models$" for team semantics instead of the symbol "$\Vdash$" which was used for Kripke semantics.

Definition 3. *Let $\mathfrak{M}$ be a Kripke model and T a team of $\mathfrak{M}$. The satisfaction relation $\mathfrak{M}, T \models \varphi$ for $\mathcal{ML}(\Phi)$ is defined as follows.*

$$\begin{aligned}
\mathfrak{M}, T \models p \quad &\Leftrightarrow \quad w \in V(p) \text{ for every } w \in T.\\
\mathfrak{M}, T \models \neg p \quad &\Leftrightarrow \quad w \notin V(p) \text{ for every } w \in T.\\
\mathfrak{M}, T \models (\varphi \wedge \psi) \quad &\Leftrightarrow \quad \mathfrak{M}, T \models \varphi \text{ and } \mathfrak{M}, T \models \psi.\\
\mathfrak{M}, T \models (\varphi \vee \psi) \quad &\Leftrightarrow \quad \mathfrak{M}, T_1 \models \varphi \text{ and } \mathfrak{M}, T_2 \models \psi \text{ for some } T_1 \text{ and } T_2\\
&\qquad \text{such that } T_1 \cup T_2 = T.\\
\mathfrak{M}, T \models \Diamond\varphi \quad &\Leftrightarrow \quad \mathfrak{M}, T' \models \varphi \text{ for some } T' \text{ such that } T[R]T'.\\
\mathfrak{M}, T \models \Box\varphi \quad &\Leftrightarrow \quad \mathfrak{M}, T' \models \varphi, \text{ where } T' = R[T].
\end{aligned}$$

A set Γ of formulas is *valid in a model* $\mathfrak{M} = (W, R, V)$ (in team semantics), in symbols $\mathfrak{M} \models \Gamma$, if $\mathfrak{M}, T \models \varphi$ holds for every team T of $\mathfrak{M}$ and every $\varphi \in \Gamma$. Likewise, we say that Γ is *valid in a Kripke frame* $\mathfrak{F}$ and write $\mathfrak{F} \models \Gamma$, if $(\mathfrak{F}, V) \models \Gamma$ hold for every valuation V. When Γ is a singleton $\{\varphi\}$, we simply write $\mathfrak{M} \models \varphi$ and $\mathfrak{F} \models \varphi$.

The formulas of $\mathcal{ML}$ have the following flatness property.

Proposition 1 (Flatness). *Let $\mathfrak{M}$ be a Kripke model and T be a team of $\mathfrak{M}$. Then, for every formula φ of $\mathcal{ML}(\Phi)$*

$$\mathfrak{M}, T \models \varphi \Leftrightarrow \forall w \in T : \mathfrak{M}, w \Vdash \varphi.$$

From flatness if follows that for every model $\mathfrak{M}$, frame $\mathfrak{F}$, and formula φ of $\mathcal{ML}$, $\mathfrak{M} \Vdash \varphi$ iff $\mathfrak{M} \models \varphi$ and $\mathfrak{F} \Vdash \varphi$ iff $\mathfrak{F} \models \varphi$.

Recall from Section 2 what it means that a set of modal formulas defines a class of frames and models. All the related definitions can be adapted for logics with team semantics by simply substituting $\Vdash$ by $\models$.

Definition 4. *We write $\mathcal{L} \leq_M \mathcal{L}'$ if every $\mathcal{L}$-definable class of models is also $\mathcal{L}'$-definable. We write $\mathcal{L} =_M \mathcal{L}'$ if both $\mathcal{L} \leq_M \mathcal{L}'$ and $\mathcal{L}' \leq_M \mathcal{L}$ hold and write $\mathcal{L} <_M \mathcal{L}'$ if $\mathcal{L} \leq_M \mathcal{L}'$ but $\mathcal{L}' \not\leq_M \mathcal{L}$.*

Definition 5. *We write $\mathcal{L} \leq_F \mathcal{L}'$ if every $\mathcal{L}$-definable class of frames is also $\mathcal{L}'$-definable. We write $\mathcal{L} =_F \mathcal{L}'$ if both $\mathcal{L} \leq_F \mathcal{L}'$ and $\mathcal{L}' \leq_F \mathcal{L}$ hold and write $\mathcal{L} <_F \mathcal{L}'$ if $\mathcal{L} \leq_F \mathcal{L}'$ but $\mathcal{L}' \not\leq_F \mathcal{L}$.*

The most important closure properties in the study of team-based logics are downward closure, union closure, and the concept of team bisimulation.

Definition 6. *Let $\mathcal{L}$ be some team-based modal logic, $\mathfrak{M}$ a Kripke model, and T, S teams of $\mathfrak{M}$. We say that a formula $\varphi \in \mathcal{L}$ is*

1. downward closed *if $\mathfrak{M}, T \models \varphi$, whenever $\mathfrak{M}, S \models \varphi$ and $T \subseteq S$.*
2. union closed if $\mathfrak{M}, T \cup S \models \varphi$, *whenever $\mathfrak{M}, T \models \varphi$ and $\mathfrak{M}, S \models \varphi$.*

A logic $\mathcal{L}$ is called downward closed *(*union closed*) if every formula $\varphi \in \mathcal{L}$ is downward closed (union closed). We say that $\mathcal{L}$ has the* empty team *property, if $\mathfrak{M}, \emptyset \models \varphi$ holds for every model $\mathfrak{M}$ and every formula $\varphi \in \mathcal{L}$.*

Team bisimulation and its finite approximation team k-bisimulation can be defined via the corresponding concepts of ordinary modal logic. In the definition below, we denote by $\rightleftarrows$ and $\rightleftarrows_k$ the notions of bisimulation and k-bisimulation of ordinary modal logic (see, e.g., [2]), respectively.

Definition 7. *Let $\mathfrak{M}, T$ and $\mathfrak{M}', T'$ be team pointed Kripke models. We say that $\mathfrak{M}, T$ and $\mathfrak{M}', T'$ are team bisimilar, and write $\mathfrak{M}, T \, [\rightleftarrows] \, \mathfrak{M}', T'$ if*

1. *for every $w \in T$ there exist some $w' \in T'$ such that $\mathfrak{M}, w \rightleftarrows \mathfrak{M}', w'$, and*
2. *for every $w' \in T'$ there exist some $w \in T$ such that $\mathfrak{M}, w \rightleftarrows \mathfrak{M}', w'$.*

The team k-bisimulation *relation $[\rightleftarrows_k]$ is defined analogously with $\rightleftarrows$ replaced by $\rightleftarrows_k$.*

4.2 Extensions of modal logic via connectives

We first introduce two expressive extensions of modal logic: an extension by the so-called intuitionistic disjunction and an extension by the so-called contradictory negation. These two logics are of great interest, since with respect to expressive power the logics subsume all most studied team-based modal logics, in particular all of those defined in Section 4.3.

Modal logic with intuitionistic disjunction $\mathcal{ML}(\varovee)(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the grammar rule $\varphi ::= (\varphi \varovee \varphi)$ with the following semantics:

$$\mathfrak{M}, T \models (\varphi \varovee \psi) \quad \Leftrightarrow \quad \mathfrak{M}, T \models \varphi \text{ or } \mathfrak{M}, T \models \psi.$$

Modal team logic $\mathcal{MTL}(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the contradictory negation, i.e., the grammar rule $\varphi ::= {\sim}\varphi$ with the following semantics:

$$\mathfrak{M}, T \models {\sim}\varphi \quad \Leftrightarrow \quad \mathfrak{M}, T \not\models \varphi.$$

The following theorem for $\mathcal{ML}(\varovee)$ was proven by Hella et al. [11] and for $\mathcal{MTL}$ by Kontinen et al. [17]

Theorem 4. *A class $\mathbb{C}$ of team pointed Kripke models is definable by a single formula of*

1. *$\mathcal{ML}(\varovee)$ iff $\mathbb{C}$ is downward closed, closed under team k-bisimulation, for some $k \in \mathbb{N}$, and admits the empty team property.*
2. *$\mathcal{MTL}$ iff $\mathbb{C}$ is closed under team k-bisimulation, for some $k \in \mathbb{N}$.*

4.3 Extensions of modal logic with atomic dependency notions

The syntax of *modal dependence logic* $\mathcal{MDL}(\Phi)$ and *extended modal dependence logic* $\mathcal{EMDL}(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the following grammar rule for each $n \in \omega$:

$$\varphi ::= \mathrm{dep}(\varphi_1, \dots, \varphi_n, \psi)\,, \text{ where } \varphi_1, \dots, \varphi_n, \psi \in \mathcal{ML}(\Phi).$$

In the additional grammar rules above for $\mathcal{MDL}$, we require that $\varphi_1, \dots, \varphi_n, \psi$ are proposition symbols in Φ. The intuitive meaning of the (modal) dependence atom $\mathrm{dep}(\varphi_1, \dots, \varphi_n, \psi)$ is that the truth value of the formula ψ is completely determined by the truth values of $\varphi_1, \dots, \varphi_n$. The formal definition is given below:

$$\begin{aligned}\mathfrak{M}, T \models \mathrm{dep}(\varphi_1, \dots, \varphi_n, \psi) \quad \Leftrightarrow \quad & \forall w, v \in T : \bigwedge_{1 \le i \le n} (\mathfrak{M}, \{w\} \models \varphi_i \Leftrightarrow \mathfrak{M}, \{v\} \models \varphi_i) \\ & \text{implies } (\mathfrak{M}, \{w\} \models \psi \Leftrightarrow \mathfrak{M}, \{v\} \models \psi).\end{aligned}$$

The syntax of *modal inclusion logic* $\mathcal{MINC}(\Phi)$ and *extended modal inclusion logic* $\mathcal{EMINC}(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the following grammar rule for each $n \in \omega$:

$$\varphi ::= \varphi_1, \dots, \varphi_n \subseteq \psi_1, \dots, \psi_n, \text{ where } \varphi_1, \psi_1, \dots, \varphi_n, \psi_n \in \mathcal{ML}(\Phi).$$

In the additional grammar rules above for $\mathcal{MINC}$, we require that the formulas $\varphi_1, \psi_1, \ldots, \varphi_n, \psi_n$ are proposition symbols in Φ. The meaning of the (modal) inclusion atom $\varphi_1, \ldots, \varphi_n \subseteq \psi_1, \ldots, \psi_n$ is that the truth values that occur in a given team for the tuple $\varphi_1, \ldots, \varphi_n$ occur also as truth values for the tuple $\psi_1, \ldots \psi_n$. The formal definition is given below:

$$\begin{aligned}\mathfrak{M}, T \models & \varphi_1, \ldots, \varphi_n \subseteq \psi_1, \ldots, \psi_n \\ & \Leftrightarrow \forall w \in T \exists v \in T : \bigwedge_{1 \leq i \leq n} (\mathfrak{M}, \{w\} \models \varphi_i \Leftrightarrow \mathfrak{M}, \{v\} \models \psi_i).\end{aligned}$$

With respect to expressive power the following are known, see e.g., [3, 12]:

$$\mathcal{ML} < \mathcal{MDL} < \mathcal{EMDL} = \mathcal{ML}(\varovee) < \mathcal{MTL}$$

$$\mathcal{ML} < \mathcal{MINC} < \mathcal{EMINC} < \mathcal{MTL}.$$

The fact that $\mathcal{MINC} < \mathcal{EMINC}$ holds is known but no published proof is known by the authors. The proof is an easy exercise, see Appendix C.

Proposition 2 (Closure properties). *The logics weaker or equal to $\mathcal{ML}(\varovee)$ with respect to expressive power are downward closed. The logics weaker or equal to $\mathcal{EMINC}$ with respect to expressive power are union closed.*

Note that the $\mathcal{MTL}$ is neither downward nor union closed. The modal depth of φ, denoted by $\mathrm{md}(\varphi)$, is defined in the obvious way (for basic modal logic, see e.g., [2]); intuitionistic disjunction and contradictory negation are handled in the same manner as Boolean connectives. For dependence atoms and inclusion atoms, we define that

$$\begin{aligned}\mathrm{md}(\mathrm{dep}(\varphi_1, \ldots, \varphi_n, \psi)) &= \max\{\mathrm{md}(\varphi_1), \ldots, \mathrm{md}(\varphi_n), \mathrm{md}(\psi)\}, \\ \mathrm{md}(\varphi_1, \ldots, \varphi_n \subseteq \psi_1, \ldots, \psi_n) &= \max\{\mathrm{md}(\varphi_1), \mathrm{md}(\psi_1), \ldots, \mathrm{md}(\varphi_n), \mathrm{md}(\psi_n)\}.\end{aligned}$$

If $\mathcal{L}$ is a logic and $k \in \mathbb{N}$, we write $\mathfrak{M}, T \equiv_k^{\mathcal{L}} \mathfrak{M}', T'$, if $\mathfrak{M}, T$ and $\mathfrak{M}', T'$ agree on all $\mathcal{L}$-formulas φ with $\mathrm{md}(\varphi) \leq k$.

Theorem 5 ([17]). *Let $\mathcal{L}$ be a team-based logic that is weaker or equal to $\mathcal{MTL}$ with respect to expressive power. Then $\mathfrak{M}, T \, [\rightleftarrows_k] \, \mathfrak{M}', T' \Rightarrow \mathfrak{M}, T \equiv_k^{\mathcal{L}} \mathfrak{M}', T'$.*

5 Modal definability in team semantics

The expressive power of the most studied team-based modal logics is quite well understood. See Table 1 for the known characterisations. However the related topics of definability with respect to models and with respect to frames has received less attention. In [18] a Goldbaltt-Thomason -style characterisation is given for modal dependence logic. Moreover it was shown that with respect to frame definability $\mathcal{MDL}$ and $\mathcal{EMDL}$ coincide. In this section we study definability of $\mathcal{MINC}$ and $\mathcal{MTL}$, see Tables 2 and 3 for a summary of known results together with the results of this sections on definability.

[4] If a class of frames is closed under disjoint unions and bounded morphic images then it reflects finitely generated subframes.

Logic	Closure properties				References
	empty team property	team k-bisimulation	downward closure	union closure	
$\mathcal{ML}$	×	×	×	×	[13]
$\mathcal{ML}(\ovee)$	×	×	×		[11, Cor. 3.6]
$\mathcal{EMDL}$	×	×	×		[11, Cor. 4.5]
$\mathcal{EMINC}$	×	×		×	[12, Thm. 3.10]
$\mathcal{MTL}$		×			[17, Thm. 3.4]

Table 1. Characterisation of expressive powers of different team-based logics. E.g., a class $\mathbb{C}$ of team pointed Kripke models is definable by a single $\mathcal{EMDL}$-formula if and only if $\mathfrak{M}, \emptyset \in \mathbb{C}$, for every $\mathfrak{M}$, $\mathbb{C}$ is closed under the so-called team k-bisimulation, for some finite k, and $\mathbb{C}$ is downward closed.

Logic	Closure under			Reflects		References
	disjoint unions	bounded morphic images	generated subframes	ultrafilter extensions	finitely generated subframes	
$\mathcal{ML}$	×	×	×	×	$\times^4$	[7]
$\mathcal{MINC}$						Theorem 8
$\mathcal{EMINC}$						Theorem 8
$\mathcal{ML}(\boxed{\uplus}^+)$						[18, Thm. 3]
$\mathcal{ML}(\ovee)$						[18, Cor. 1]
$\mathcal{MDL}$		×	×	×	×	[18, Cor. 1]
$\mathcal{EMDL}$						[18, Cor. 1]
$\mathcal{MTL}$						Theorem 11
$\mathcal{ML}(\boxed{\uplus})$		×		×		[9, Cor. 3.9]

Table 2. Characterisation of frame definability of different modal logics with respect to first-order definable frame classes. E.g., an elementary class $\mathbb{F}$ of Kripke frames is definable in $\mathcal{EMDL}$ if and only if $\mathbb{F}$ is closed under taking generated subframes and bounded morphic images, and reflects ultrafilter extensions and finitely generated subframes.

Model definability	$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} <_M \mathcal{MDL} <_M \{\mathcal{EMDL}, \mathcal{ML}(\ovee), \mathcal{MTL}\}$
Frame definability	$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} <_F \{\mathcal{MDL}, \mathcal{EMDL}, \mathcal{ML}(\ovee), \mathcal{MTL}\}$

Table 3. Hierarchy of definability of different modal logics with team semantics.

5.1 Hintikka formulas and types

It is well known that for any finite set of proposition symbols Φ, any finite $k \in \mathbb{N}$, and any pointed Φ-model (K, w), there exists a modal formula of modal depth k that characterises (K, w) completely up to k-equivalence (i.e. equivalence up to modal depth k). These *Hintikka formulas* (or *characteristic formulas*) are defined as follows (see e.g. [8]):

Definition 8. *Assume that Φ is a finite set of proposition symbols. Let $k \in \mathbb{N}$ and let $(\mathfrak{M}, w)$ be a pointed Φ-model. The k-th* Hintikka formula $\chi^k_{\mathfrak{M},w}$ *of $(\mathfrak{M}, w)$ is defined recursively as follows:*

- $\chi^0_{\mathfrak{M},w} := \bigwedge\{p \mid p \in \Phi, w \in V(p)\} \wedge \bigwedge\{\neg p \mid p \in \Phi, w \notin V(p)\}$.
- $\chi^{k+1}_{\mathfrak{M},w} := \chi^k_{\mathfrak{M},w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^k_{\mathfrak{M},v} \wedge \Box \bigvee_{v \in R[w]} \chi^k_{\mathfrak{M},v}$.

It is easy to see that $\mathrm{md}\Big(\chi^k_{\mathfrak{M},w}\Big) = k$, and $\mathfrak{M}, w \models \chi^k_{\mathfrak{M},w}$ for every pointed Φ-model $(\mathfrak{M}, w)$. By a straightforward inductive argument, it can be shown that, for each finite Φ and k, there are only finitely many non-equivalent k-th Hintikka formulas. Thus $\chi^k_{\mathfrak{M},w}$ is essentially finite (the possibly infinite conjunction $\bigwedge_{v \in R[w]}$ and disjunction $\bigvee_{v \in R[w]}$ can be replaced by finite ones while preserving equivalence).

Proposition 3 (see, e.g., [8]). *Let Φ be a finite set of proposition symbols, $k \in \mathbb{N}$, and $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ pointed Φ-models. Then*

$$\mathfrak{M}, w \equiv^{\mathcal{ML}}_k \mathfrak{M}', w' \quad \iff \quad \mathfrak{M}', w' \models \chi^k_{\mathfrak{M},w}.$$

Definition 9. *Let $\mathfrak{M}$ be a Kripke Φ-model and $\mathbb{C}$ a class of Kripke Φ-models. We define that*

$$\begin{aligned}
\mathrm{tp}^\Phi_k(\mathfrak{M}) &:= \{\chi^k_{\mathfrak{M},w} \mid w \text{ is a point of } \mathfrak{M}\}, \\
\mathrm{tp}^\Phi_k(\mathfrak{M}, T) &:= \{\chi^k_{\mathfrak{M},w} \mid w \in T\}, \\
\mathrm{tp}^\Phi_k(\mathbb{C}) &:= \{\mathrm{tp}^\Phi_k(\mathfrak{M}) \mid \mathfrak{M} \in \mathbb{C}\}.
\end{aligned}$$

Proposition 4. *Let $\mathcal{L}$ be any team-based logic weaker than or equal to $\mathcal{MTL}$ w.r.t. expressive power. Then $\mathrm{tp}^\Phi_k(\mathfrak{M}, T) = \mathrm{tp}^\Phi_k(\mathfrak{M}', T') \Rightarrow \mathfrak{M}, T \equiv^{\mathcal{L}}_k \mathfrak{M}', T'$.*

Proof. Assume that $\mathrm{tp}^\Phi_k(\mathfrak{M}, T) = \mathrm{tp}^\Phi_k(\mathfrak{M}', T')$. By Proposition 3 and the definition of team bisimulation, it follows that $\mathfrak{M}, T \; [\rightleftarrows_k] \; \mathfrak{M}', T'$. The claim now follows by Theorem 5. $\Box$

5.2 Global modal & frame definability in $\mathcal{MINC}$ and $\mathcal{ML}$ coincide

Lemma 2. *Let Φ be a finite set of proposition symbols, $\varphi \in \mathcal{EMINC}(\Phi)$, and $k = \mathrm{md}(\varphi)$. Then $\mathfrak{M} \in \mathrm{Mod}(\varphi)$ iff $\mathrm{tp}^\Phi_k(\mathfrak{M}) \subseteq \bigcup\{\mathrm{tp}^\Phi_k(\mathfrak{M}') \mid \mathfrak{M}' \in \mathrm{Mod}(\varphi)\}$.*

Proof. The direction from left to right is trivial. Assume then that

$$\mathrm{tp}_k^{\Phi}(\mathfrak{M}) \subseteq \bigcup \{\mathrm{tp}_k^{\Phi}(\mathfrak{M}') \mid \mathfrak{M}' \in \mathrm{Mod}(\varphi)\} \tag{1}$$

holds, and let T be an arbitrary team of $\mathfrak{M}$. It suffices to establish that $\mathfrak{M}, T \models \varphi$. From (1) it follows that there exists some $n \in \mathbb{N}$, models $\mathfrak{M}_i \in \mathrm{Mod}(\varphi)$, teams S_i of $\mathfrak{M}_i$ and T_i of $\mathfrak{M}$, $i \leq n$, such that

$$T_1 \cup \cdots \cup T_n = T \text{ and } \mathrm{tp}_k^{\Phi}(\mathfrak{M}_i, S_i) = \mathrm{tp}_k^{\Phi}(\mathfrak{M}, T_i), \text{ for each } i \leq n.$$

Note that such finite n exists, since $\mathrm{tp}_k^{\Phi}(\mathfrak{M})$ is essentially finite. Since each $\mathfrak{M}_i \in \mathrm{Mod}(\varphi)$, it follows that $\mathfrak{M}_i, S_i \models \varphi$, for $i \leq n$. Thus from Proposition 4 and the fact that $\mathrm{tp}_k^{\Phi}(\mathfrak{M}_i, S_i) = \mathrm{tp}_k^{\Phi}(\mathfrak{M}, T_i)$, for $i \leq n$, it follows that $\mathfrak{M}, T_i \models \varphi$, for $i \leq n$. Now, by union closure (Proposition 2), we conclude that $\mathfrak{M}, T \models \varphi$. □

Theorem 6. *A class $\mathbb{C}$ of Kripke models is definable by a single $\mathcal{EMINC}$-formula if and only if the class if definable by a single $\mathcal{ML}$-formula.*

Proof. The if direction is trivial. For the other direction, let $\mathbb{C}$ be a class of Kripke models that is definable by a single $\mathcal{EMINC}$ formula and let φ be an $\mathcal{EMINC}(\Phi)$-formula that defines $\mathbb{C}$. Without lose of generality, we may assume that Φ is finite. Let k denote the modal depth of φ. We will show that the $\mathcal{ML}(\Phi)$ formula

$$\varphi^* := \bigvee \{\chi^k_{\mathfrak{M},w} \mid \mathfrak{M} \in \mathrm{Mod}(\varphi), w \in \mathfrak{M}\}$$

defines $\mathbb{C}$. Since over a finite set of proposition symbols there exists only finitely many essentially different k-Hintikka-formulas, φ^* is essentially a finite $\mathcal{ML}(\Phi)$ -formula. By assumption $\mathbb{C} = \mathrm{Mod}(\varphi)$. Thus by Lemma 2

$$\mathfrak{M} \in \mathbb{C} \text{ iff } \mathrm{tp}_k^{\Phi}(\mathfrak{M}) \subseteq \bigcup \{\mathrm{tp}_k^{\Phi}(\mathfrak{M}') \mid \mathfrak{M}' \in \mathrm{Mod}(\varphi)\}. \tag{2}$$

Observe that by flatness (Proposition 1)

$$\mathfrak{M}, T \models \varphi^* \text{ iff } \mathrm{tp}_k^{\Phi}(\mathfrak{M}, T) \subseteq \bigcup \{\mathrm{tp}_k^{\Phi}(\mathfrak{M}') \mid \mathfrak{M}' \in \mathrm{Mod}(\varphi)\},$$

and thus it follows that

$$\mathfrak{M} \models \varphi^* \text{ iff } \mathrm{tp}_k^{\Phi}(\mathfrak{M}) \subseteq \bigcup \{\mathrm{tp}_k^{\Phi}(\mathfrak{M}') \mid \mathfrak{M}' \in \mathrm{Mod}(\varphi)\}. \tag{3}$$

From (2) and (3) the claim follows. □

The following theorems directly follow.

Theorem 7. *$\mathcal{EMINC} =_M \mathcal{ML}$.*

Theorem 8. *$\mathcal{EMINC} =_F \mathcal{ML}$.*

Proof. Clearly any $\mathcal{ML}$-definable class of Kripke frames is also definable in $\mathcal{EMINC}$. The converse follows directly from Theorem 6.

Let $\mathfrak{F}$ be a Kripke frame, φ an $\mathcal{EMINC}$-formula and φ^* the related $\mathcal{ML}$-formula given by Theorem 6 such that $\mathrm{Mod}(\varphi) = \mathrm{Mod}(\varphi^*)$. Now, by definition, $\mathfrak{F} \models \varphi$ if and only if $(\mathfrak{F}, V) \models \varphi$ for every valuation V. Since $\mathrm{Mod}(\varphi) = \mathrm{Mod}(\varphi^*)$, this holds if and only if $(\mathfrak{F}, V) \models \varphi^*$ for every valuation V, which by definition holds if and only if $\mathfrak{F} \models \varphi^*$. Now let $\mathbb{F}$ be some $\mathcal{EMINC}$-definable class of Kripke frames and let Γ be a set of $\mathcal{EMINC}$-formulas that defines $\mathbb{F}$. Define $\Gamma^* := \{\varphi^* \mid \varphi \in \Gamma\}$. Clearly Γ^* is a set of $\mathcal{ML}$-formulas that defines $\mathbb{F}$. $\square$

5.3 Global modal & frame definability in $\mathcal{MTL}$ & $\mathcal{ML}(\otimes)$ coincide

Lemma 3. *Let φ be and $\mathcal{MTL}$-formula and $k = \mathrm{md}(\varphi)$. Then*

$$\mathfrak{M} \in \mathrm{Mod}(\varphi) \textit{ iff } \mathrm{tp}_k^{\Phi}(\mathfrak{M}) \subseteq \Gamma \in \mathrm{tp}_k^{\Phi}\big(\mathrm{Mod}(\varphi)\big), \textit{ for some } \Gamma.$$

Proof. The direction from left to right is trivial. Assume then that $\mathrm{tp}_k^{\Phi}(\mathfrak{M}) \subseteq \Gamma \in \mathrm{tp}_k^{\Phi}\big(\mathrm{Mod}(\varphi)\big)$ holds for some Γ. Thus there exists a Kripke model $\mathfrak{M}'$ such that $\mathfrak{M}' \in \mathrm{Mod}(\varphi)$ and $\mathrm{tp}_k^{\Phi}(\mathfrak{M}') = \Gamma$. For the sake of a contradiction, assume that $\mathfrak{M} \notin \mathrm{Mod}(\varphi)$. Thus there exists a team T of $\mathfrak{M}$ such that $\mathfrak{M}, T \not\models \varphi$. Since $\mathrm{tp}_k^{\Phi}(\mathfrak{M}) \subseteq \mathrm{tp}_k^{\Phi}(\mathfrak{M}')$ it follows that there exists a team T' of $\mathfrak{M}'$ such that $\mathrm{tp}_k^{\Phi}(\mathfrak{M}, T) = \mathrm{tp}_k^{\Phi}(\mathfrak{M}', T')$. Thus by Proposition 4, we conclude that $\mathfrak{M}', T' \not\models \varphi$. This is a contradiction and thus $\mathfrak{M} \in \mathrm{Mod}(\varphi)$ holds. $\square$

Theorem 9. *A class $\mathbb{C}$ of Kripke models is definable in $\mathcal{MTL}$ by a single formula if and only if it is definable in $\mathcal{ML}(\otimes)$ by a single formula.*

Proof. The fact that every class of Kripke models that is definable by a single $\mathcal{ML}(\otimes)$-formula is also definable by a single $\mathcal{MTL}$-formula follows directly by Theorem 4.

Let $\mathbb{C}$ be an arbitrary single formula $\mathcal{MTL}$-definable class of Kripke models and let φ be an $\mathcal{MTL}$-formula that defines $\mathbb{C}$. Let k denote the modal depth of φ. We will show that the $\mathcal{ML}(\otimes)$-formula

$$\varphi^* := \bigotimes_{\Gamma \in \mathrm{tp}_k^{\Phi}(\mathbb{C})} \big(\bigvee \Gamma \big)$$

defines $\mathbb{C}$. Note that since $\mathrm{tp}_k^{\Phi}(\mathbb{C})$ is a family of sets of k-Hintikka formulas the outer disjunction is essentially finite. Likewise, since each Γ is a collection of k-Hintikka formulas, it follows by flatness (remember that Hintikka formulas are $\mathcal{ML}$-formulas) that the inner disjunctions are essentially finite. Thus φ^* is essentially a finite $\mathcal{ML}(\otimes)$-formula.

Assume first that $\mathfrak{M} \in \mathbb{C}$. By definition $\mathrm{tp}_k^{\Phi}(\mathfrak{M}) \in \mathrm{tp}_k^{\Phi}(\mathbb{C})$. Clearly, for each team T of $\mathfrak{M}$, it holds that $\mathfrak{M}, T \models \bigvee \mathrm{tp}_k^{\Phi}(\mathfrak{M})$, and thus that $\mathfrak{M}, T \models \varphi^*$. Therefore $\mathfrak{M} \models \varphi^*$. Assume then that $\mathfrak{M} \models \varphi^*$. Thus $\mathfrak{M}, W \models \varphi^*$, where W is the domain of $\mathfrak{M}$. Therefore there exists a set $\Gamma \in \mathrm{tp}_k^{\Phi}(\mathbb{C})$ such that $\mathfrak{M}, W \models \bigvee \Gamma$. Thus $\mathrm{tp}_k^{\Phi}(\mathfrak{M}) = \mathrm{tp}_k^{\Phi}(\mathfrak{M}, W) \subseteq \Gamma$. Recall that $\mathbb{C} = \mathrm{Mod}(\varphi)$. Now since $\Gamma \in \mathrm{tp}_k^{\Phi}(\mathbb{C}) = \mathrm{tp}_k^{\Phi}\big(\mathrm{Mod}(\varphi)\big)$, it follows from Lemma 3 that $\mathfrak{M} \in \mathrm{Mod}(\varphi) = \mathbb{C}$. $\square$

The following theorems directly follow.

Theorem 10. *$\mathcal{MTL} =_M \mathcal{ML}(\varovee)$.*

Theorem 11. *$\mathcal{MTL} =_F \mathcal{ML}(\varovee)$.*

Proof. The fact that every $\mathcal{ML}(\varovee)$ definable class of Kripke frames is also definable in $\mathcal{MTL}$ follows directly by Theorem 4.

Let $\mathbb{F}$ be an arbitrary $\mathcal{MTL}$-definable class of Kripke frames and let Γ a set of $\mathcal{MTL}$-formulas that defines $\mathbb{F}$. By Theorem 9, for each $\varphi \in \mathcal{MTL}$ there exists a formula $\varphi^* \in \mathcal{ML}(\varovee)$ such that $\mathrm{Mod}(\varphi^*) = \mathrm{Mod}(\varphi)$. Recall that φ defines the class $\mathrm{Mod}(\varphi)$ of Kripke models. Now clearly $\mathfrak{F} \models \varphi$ iff $(\mathfrak{F}, V) \in \mathrm{Mod}(\varphi)$ for every valuation V iff $(\mathfrak{F}, V) \in \mathrm{Mod}(\varphi^*)$ for every valuation V iff $\mathfrak{F} \models \varphi^*$. Define $\Gamma := \{\varphi^* \mid \varphi \in \Gamma\}$. Clearly for each frame $\mathfrak{F}$ it holds that $\mathfrak{F} \models \Gamma$ iff $\mathfrak{F} \models \Gamma^*$. □

It was established in [18] that $\mathcal{ML} <_F \mathcal{MDL} =_F \mathcal{EMDL} =_F \mathcal{ML}(\varovee)$. When combined with Theorems 8 and 11 the following hierarchy is obtained.

Theorem 12. *$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} <_F \{\mathcal{MDL}, \mathcal{EMDL}, \mathcal{ML}(\varovee), \mathcal{MTL}\}$.*

It is an easy exercise to show that $\mathcal{ML} <_M \mathcal{MDL}$ and $\mathcal{MDL} <_M \mathcal{EMDL}$, see Appendix C. Moreover it follows from the work of Hella et al. [11] that $\mathcal{EMDL} = \mathcal{ML}(\varovee)$. Thus by Theorems 7 and 10 we obtain the following trichotomy.

Theorem 13.

$$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} <_M \mathcal{MDL} <_M \{\mathcal{EMDL}, \mathcal{ML}(\varovee), \mathcal{MTL}\}.$$

6 Conclusion

In this paper, we studied relative frame definability of a fragment of modal logic with universal modality in which the universal modality occurs only positively. Moreover we studied definability of particular modal logics with team semantics. We showed that a class $\mathbb{F}$ of finite transitive frames is definable in $\mathcal{ML}(\boxed{\sqcup}^+)$ relative to finite transitive frames if and only if $\mathbb{F}$ is closed under taking generated subframes and bounded morphic images. In addition, we established the following trichotomy with respect to model definability

$$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} <_M \mathcal{MDL} <_M \{\mathcal{EMDL}, \mathcal{ML}(\varovee), \mathcal{MTL}\}$$

and the following dichotomy with respect to frame definability

$$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} <_F \{\mathcal{MDL}, \mathcal{EMDL}, \mathcal{ML}(\varovee), \mathcal{MTL}\}.$$

Since it is known that $\mathcal{MDL} =_F \mathcal{ML}(\boxed{\sqcup}^+)$, we obtained as a corollary relative Goldblatt–Thomason -style theorems for each of the logics listed above.

Note that our results imply that the model (frame) definability of every logic between $\mathcal{EMDL}$ ($\mathcal{MDL}$) and $\mathcal{MTL}$ coincides. In particular, we obtain results

concerning modal independence logic $\mathcal{MIL}$ and extended modal independence logic $\mathcal{EMIL}$, since with respect to expressive power $\mathcal{MDL} \leq \mathcal{MIL} \leq \mathcal{MTL}$ and $\mathcal{EMDL} \leq \mathcal{EMIL} \leq \mathcal{MTL}$.

We conclude with some open questions:

1. Where does $\mathcal{MIL}$ lie with respect to modal definability?
2. Is there some natural fragment of $\mathcal{ML}(\boxdot^{+})$ that coincided with $\mathcal{MDL}$ or $\mathcal{MIL}$ with respect to model definability?
3. Can we use the notion of local bounded morphism (cf. [1]) to drop the requirement of transitivity from Theorem 3.
4. Can we characterize model definability of team-based logics in terms of semantic constructions?

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A Proof of Lemma 1

Lemma 1. *Let $\mathfrak{F} = (W, R)$ be a finite transitive frame. For any transitive frame $\mathfrak{G}$, the following are equivalent:*

(i) *the Jankov-Fine formula $\varphi_{\mathfrak{F}}$ is not valid in $\mathfrak{G}$,*
(ii) *there is a finite set $Y \subseteq |\mathfrak{G}|$ such that $\mathfrak{F}$ is a bounded morphic image of $\mathfrak{G}_Y$, where $\mathfrak{G}_Y$ is the subframe of $\mathfrak{G}$ generated by Y.*

Proof. The direction from (ii) to (i) is immediate from the fact that $\varphi_{\mathfrak{F}}$ is not valid in $\mathfrak{F}$ under the natural valuation sending p_{w_i} to $\{ w_i \}$ (Note: validity of $\mathcal{ML}(\boxed{\cup}^+)$-formulas is closed under taking under bounded morphic images and generated subframes, see [18]). So, we focus on the converse direction.

Assume (i). It follows from $\mathfrak{G} \not\Vdash \varphi_{\mathfrak{F}}$ that $(\mathfrak{G}, V) \not\Vdash \varphi_{\mathfrak{F}}$, for some assignment V. Thus, for each $i \leq n$, there exists a point v_i of $\mathfrak{G}$ such that $(\mathfrak{G}, V), v_i \Vdash \varphi_{\mathfrak{F},w_i}$. Put $Y := \{ v_i \,|\, 0 \leq i \leq n \}$, let $\mathfrak{G}_Y$ denote the subframe of $\mathfrak{G}$ generated by Y, and let U be the reduction of V into the frame $\mathfrak{G}_Y$. Since satisfaction of $\mathcal{ML}$-formulas is closed under taking generated submodels (see, e.g., [2, Prop. 2.6]), it follows that $(\mathfrak{G}_Y, U), v_i \Vdash \varphi_{\mathfrak{F},w_i}$, for each $i \leq n$. Let us put $\mathfrak{G}_Y = (G_Y, S)$. The first clause of the Jankov-Fine formula $\varphi_{\mathfrak{F},w_i}$ implies that, for each $i \leq n$, $U(p_{w_i}) \neq \emptyset$. By the second and the third clause, we obtain $\bigcup_{w \in W} U(p_w) = G_Y$ and $U(p_{w_i}) \cap U(p_{w_j}) = \emptyset$ for any distinct indices i and j. This enables us to define a surjective mapping $f : G_Y \to W$. Define $f(v) := w_i$ if $v \in U(p_{w_i})$. Clearly f is a well defined surjection.

In what follows, we show that f is a bounded morphism. The condition (Forth) is established as follows. Assume that xSy and let i, j be such that $f(x) = w_i$ and $f(y) = w_j$. Thus $x \in U(p_{w_i})$ and $y \in U(p_{w_j})$. Since $\mathfrak{G}_Y$ is Y-generated, x is reachable from some $v_k \in Y$. Suppose for a contradiction that $w_i R w_j$ fails in $\mathfrak{F}$. Then $\Box^+(p_{w_i} \to \neg\Diamond p_{w_j})$ is a conjunct in the Jankov-Fine formula $\varphi_{\mathfrak{F},w_k}$. Recall

that $(\mathfrak{G}_Y, U), v_k \Vdash \varphi_{\mathfrak{F},w_k}$. It now follows from $(\mathfrak{G}_Y, U), v_k \Vdash \Box^+(p_{w_i} \to \neg\Diamond p_{w_j})$ that xSy fails. A contradiction. Therefore, $w_i R w_j$ holds in $\mathfrak{F}$.

The condition (Back) is shown as follows. Assume that $f(x)Rw_j$ and let i be such that $f(x) = w_i$. From the definition of f, it follows that $x \in U(p_{w_i})$. Since $\mathfrak{G}_Y$ is Y-generated, x is reachable from some $v_k \in Y$. Since $w_i R w_j$, we have that $\Box^+(p_{w_i} \to \Diamond p_{w_j})$ is a conjunct in the Jankov-Fine formula $\varphi_{\mathfrak{F},w_k}$. Recall again that $(\mathfrak{G}_Y, U), v_k \Vdash \varphi_{\mathfrak{F},w_k}$. It follows from $(\mathfrak{G}_Y, U), v_k \Vdash \Box^+(p_{w_i} \to \Diamond p_{w_j})$ and $x \in U(p_{w_i})$ that there is some y such that $f(y) = w_j$ and xSy holds, as desired. $\Box$

B Frame constructions

Definition 10 (Disjoint Unions). *Let $\{ \mathfrak{F}_i \mid i \in I \}$ be a pairwise disjoint family of frames, where $\mathfrak{F}_i = (W_i, R_i)$. The* disjoint union *$\biguplus_{i\in I} \mathfrak{F}_i = (W, R)$ of $\{ \mathfrak{F}_i \mid i \in I \}$ is defined by $W = \bigcup_{i\in I} W_i$ and $R = \bigcup_{i\in I} R_i$.*

Definition 11 (Generated Subframes). *Given any two frames $\mathfrak{F} = (W, R)$ and $\mathfrak{F} = (W', R')$, $\mathfrak{F}'$ is a* generated subframe *of $\mathfrak{F}$ if* (i) *$W' \subseteq W$,* (ii) *$R' = R \cap (W')^2$,* (iii) *$w'Rv'$ implies $v' \in W'$, for every $w' \in W'$. We say that $\mathfrak{F}'$ is the* generated subframe *of $\mathfrak{F}$ by $X \subseteq |\mathfrak{F}|$ (notation: $\mathfrak{F}_X$) if $\mathfrak{F}'$ is the smallest generated subframe of $\mathfrak{F}$ whose domain contains X. $\mathfrak{F}'$ is a* finitely generated subframe *of $\mathfrak{F}$ if there is a finite set $X \subseteq |\mathfrak{F}|$ such that $\mathfrak{F}'$ is $\mathfrak{F}_X$.*

A frame class $\mathbb{F}$ *reflects* finitely generated subframes whenever it is the case for all frames $\mathfrak{F}$ that, if every finitely generated subframe of $\mathfrak{F}$ is in $\mathbb{F}$, then $\mathfrak{F} \in \mathbb{F}$.

Definition 12 (Bounded Morphism). *Given any two frames $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$, a function $f : W \to W'$ is a* bounded morphism *if it satisfies the following two conditions:*

(**Forth**) *If wRv, then $f(w)R'f(v)$.*
(**Back**) *If $f(w)R'v'$, then wRv and $f(v) = v'$ for some $v \in W$.*

If f is surjective, we say that $\mathfrak{F}'$ is a bounded morphic image *of $\mathfrak{F}$.*

Definition 13 (Ultrafilter Extensions). *Let $\mathfrak{F} = (W, R)$ be a Kripke frame, and $\mathrm{Uf}(W)$ denote the set of all ultrafilters on W. Define the binary relation $R^{\mathfrak{ue}}$ on the set $\mathrm{Uf}(W)$ as follows: $\mathcal{U} R^{\mathfrak{ue}} \mathcal{U}'$ iff $X \in \mathcal{U}'$ implies $m_R(X) \in \mathcal{U}$, for every $X \subseteq W$, where $m_R(X) := \{ w \in W \mid wRw'$ for some $w' \in X \}$. The frame $\mathfrak{ue}\mathfrak{F} = (\mathrm{Uf}(W), R^{\mathfrak{ue}})$ is called the* ultrafilter extension *of $\mathfrak{F}$.*

A frame class $\mathbb{F}$ *reflects* ultrafilter extensions if $\mathfrak{ue}\mathfrak{F} \in \mathbb{F}$ implies $\mathfrak{F} \in \mathbb{F}$ for every frame $\mathfrak{F}$.

C Separations in definability

Proposition 5. *With respect to expressive power $\mathcal{MINC} < \mathcal{EMINC}$.*

Proof. For $\varphi \in \mathcal{MINC}(\{p\})$, let φ^* denote the $\mathcal{ML}(\{p\})$-formula obtained from φ by substituting each inclusion atom in φ by the formula $(p \vee \neg p)$. Since $p \subseteq p$ is essentially the only inclusion atom in $\mathcal{MINC}(\{p\})$, it is easy to see that, for every $\varphi \in \mathcal{MINC}(\{p\})$, φ and φ^* are equivalent.

Let $\mathfrak{M} = (W, R, V)$ be a Kripke $\{p\}$-model such that $W = \{1, 2, 3\}$, R={(1,2)}, and $V(p) = \{1, 2, 3\}$. We claim that there does not exists a $\mathcal{MINC}$-formula that is equivalent with $p \subseteq \Diamond p$. For the sake of a contradiction, assume that $\psi \in \mathcal{MINC}$ is such a formula. Clearly $\mathfrak{M}, \{1, 3\} \models p \subseteq \Diamond p$ and thus, by assumption, $\mathfrak{M}, \{1, 3\} \models \psi$. By our observation above, $\mathfrak{M}, \{1, 3\} \models \psi^*$ follows. Now since ψ^* is an $\mathcal{ML}$-formula, it follows by Proposition 1 that $\mathfrak{M}, \{3\} \models \psi^*$. Thus $\mathfrak{M}, \{3\} \models \psi$ and therefore $\mathfrak{M}, \{3\} \models p \subseteq \Diamond p$. However clearly $\mathfrak{M}, \{3\} \not\models p \subseteq \Diamond p$, a contradiction.

Proposition 6. *$\mathcal{ML} <_M \mathcal{MDL}$.*

Proof. Let $\mathfrak{M}_i = (W_i, R_i, V_i)$, $i \leq 2$, be Φ-models such that $W_0 = \{1, 2\}$, $W_1 = \{1\}$, $W_2 = \{2\}$, $R_0 = R_1 = R_2 = \emptyset$, and, for each $p \in \Phi$, $V_0(p) = V_1(p) = \{1\}$, and $V_2(p) = \emptyset$. It is easy to conclude by flatness of $\mathcal{ML}$ that

$$\mathfrak{M}_0 \in \mathrm{Mod}(\varphi) \text{ iff } \mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Mod}(\varphi)$$

holds for every $\varphi \in \mathcal{ML}$. Thus

$$\mathfrak{M}_0 \in \mathrm{Mod}(\Gamma) \text{ iff } \mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Mod}(\Gamma)$$

holds for every $\Gamma \subseteq \mathcal{ML}$. However $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Mod}(\mathrm{dep}(p))$ but $\mathfrak{M}_0 \notin \mathrm{Mod}(\mathrm{dep}(p))$. Thus we conclude that $\mathrm{Mod}(\mathrm{dep}(p))$ is not definable in $\mathcal{ML}$.

Proposition 7. *$\mathcal{MDL} <_M \mathcal{EMDL}$.*

Proof. Let $\mathfrak{M}_i = (W_i, R_i, V_i)$, $i \leq 2$, be Φ-models such that $W_0 = \{1, 2\}$, $W_1 = \{1\}$, $W_2 = \{2\}$, $R_0 = \{(1, 1)\}$, $R_1 = \{(1, 1)\}$, $R_2 = \emptyset$, and, for each $p \in \Phi$, $V_0(p) = \{1, 2\}$, $V_1(p) = \{1\}$, and $V_2(p) = \{2\}$. It is easy to conclude (see [4, Theorem 1] for details) that

$$\mathfrak{M}_0 \in \mathrm{Mod}(\varphi) \text{ iff } \mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Mod}(\varphi)$$

holds for every $\varphi \in \mathcal{MDL}$. Thus

$$\mathfrak{M}_0 \in \mathrm{Mod}(\Gamma) \text{ iff } \mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Mod}(\Gamma)$$

holds for every $\Gamma \subseteq \mathcal{MDL}$. However $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Mod}(\mathrm{dep}(\Diamond p))$ but $\mathfrak{M}_0 \notin \mathrm{Mod}(\mathrm{dep}(\Diamond p))$. Thus we conclude that $\mathrm{Mod}(\mathrm{dep}(\Diamond p))$ is not definable in $\mathcal{MDL}$.